Week 3 2023

Thursday, February 2, 2023 6:21 PM

This week:

- finish antiderivative review
- continue w $s \leftrightarrow v \leftrightarrow a$ relationship
- explore Fundamental Theorem of Calculus (FTC)

WebAssign – Week 3
General Antiderivative

$\Updownarrow$

treat like indefinite integral (no bounds on integral sign)

+C vs C+

. . . .

Section 4.8 Antiderivatives

$\longrightarrow$ Power Functions

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C}$$

$$\int ax^n\,dx = a\int x^n\,dx$$

$$\boxed{\int a\cdot f(x)\,dx = a\int f(x)\,dx}$$

ex. given $f''(x) = 6x^1 + 12x^2$
find $f(x)$

$$\rightarrow f'(x) = \frac{6x^2}{2} + \frac{12x^3}{3} + C \quad \text{where } C \text{ is a constant}$$

$$= 3x^2 + 4x^3 + Cx^0 \qquad x^0 = 1$$

4 is deriv,
then 4x was f

$$f(x) = \frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{4}x^4}{\cancel{4}} + Cx + D$$

where D is a constant

$$\boxed{f(x) = x^3 + x^4 + Cx + D}$$

Antiderivatives with Initial Conditions

ex. given $f'(x) = 1 - 6x$
and $f(0) = 8$ ← initial condition

$$f(x) = x - \frac{\overset{3}{\cancel{6}}x^2}{\cancel{2}} \boxed{+C}$$

$$f(0) = 8 = 0 + 3(0) + C \Rightarrow C = \boxed{8}$$

$$\boxed{f(x) = x - 3x^2 + 8}$$

ex. a particle moves in a straight line
acceleration: $a(t) = 6t + 4$ cm

$a(t) = s''(t)$

ex: a particle moves in a straight line

acceleration: $a(t) = 6t + 4$ $\qquad a(t) = s''(t)$

$= v'(t)$

initial velocity: $v(0) = -6 \ \frac{cm}{s}$

initial displacement: $s(0) = 9$ cm ←

find particle's position function, $s(t)$

$$a(t) = 6t + 4$$

$$v(t) = \frac{\overset{3}{\cancel{6}}t^2}{\cancel{2}} + 4t + C$$

$$v(0) = 3(0) + 4(0) + C = -6 \Rightarrow C = -6$$

$$v(t) = 3t^2 + 4t - 6$$

$$s(t) = \frac{\cancel{3}t^3}{\cancel{3}} + \frac{4t^2}{2} - 6t + D \longrightarrow \boxed{s(t) = t^3 + 2t^2 - 6t + 9}$$

$$s(0) = \quad \cdot \quad \cdot \quad \cdot + D = 9$$

$\tan^{-1} x$

$(\arctan x)' = \frac{1}{1+x^2}$

Do: $\int_0^2 \left(2x^3 - 6x^1 + \frac{3}{1+x^2}\right) dx$

$$= \left(2\frac{x^4}{4} - 6\frac{x^2}{2}\right)\Big|_0^2 + 3\underline{\int \frac{1}{1+x^2}\,dx}$$

$$= \underbrace{\left[\frac{x^4}{2} - 3x^2 + 3\arctan x\right]_0^2}_{F(x)} = F(b) - F(a)$$

$$= F(2) - F(0)$$

$\tan 0 = 0$

$$= \frac{1}{2}(2^4 - \cancel{0^4}) - 3(2^2 - \cancel{0^2}) + 3(\arctan 2 - \arctan 0)$$

$$= \frac{1}{2} \cdot 16 \quad - 3(4) \quad + 3(\arctan 2 - 0)$$

$$= 8 \quad - 12 \quad + 3\arctan 2 = -4 + 3\arctan 2$$

$$= 3\arctan 2 - 4$$

before integrating,

may need to re-format function:

$\frac{a+b-c}{d} = \frac{a}{d} + \frac{b}{d} - \frac{c}{d}$ ←

ex. $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$

$\frac{\cancel{t^2}\sqrt{t}}{\cancel{t^2}} = \sqrt{t} = t^{1/2}$

Do: write function as 3 separate fractions

eliminate fractions and radical

$$= \int_1^9 (2 + t^{1/2} - t^{-2})\, dt$$

$$\int_a^b t^n\, dt = \frac{t^{n+1}}{n+1}\Big|_a^b$$

Do: integrate

prepare for $F(b) - F(a)$

$$= \left(2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\right)\Big|_1^9$$

$$= \left(2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\right)\Big|_1^9$$

now $F(b) - F(a)$

$x^{ab} = (x^a)^b$

$$= 2t\Big|_1^9 + \frac{2}{3}t^{3/2}\Big|_1^9 + \frac{1}{t}\Big|_1^9$$

$9^{3/2} = (9^{1/2})^3$ $\qquad \frac{3}{2} = \frac{1}{2} \cdot 3$

$$= 2(9-1) + \frac{2}{3}(9^{3/2} - 1^{3/2}) + \frac{1}{9} - \frac{1}{1}$$

$(\ )^3$

$= 2(9-1) + \frac{2}{3}(9^{3/2} - 1^{3/2}) + \frac{1}{9} - \frac{1}{1}$

$= 2(8) + \frac{2}{3}(\underbrace{27-1}_{26}) + \frac{1}{9} - 1$

$= \frac{9\cdot 15}{9}\cancel{16} + \frac{52}{3}\cdot\frac{3}{3} + \frac{1}{9}$ ← need LCD = 9

~~$F(b) - F(a)$:~~

$= \frac{135 + 156 + 1}{9} = \boxed{\frac{292}{9}}$

$\begin{array}{r} 26 \\ \times 2 \\ \hline 52 \\ \times 3 \\ \hline 156 \end{array}$

$= (\sqrt{9})^3$

$= 3^3 = 27 = 9^{3/2}$

$\begin{array}{r} 15 \\ 9 \\ \hline 135 \end{array}$

ex. $\int_0^{\sqrt{2}/2} \frac{4}{\sqrt{1-x^2}}\,dx = \textcircled{4}\underbrace{\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx}$

$= 4 \cdot \arcsin x \Big|_0^{\sqrt{2}/2}$

$= 4\left(\arcsin\frac{\sqrt{2}}{2} - \arcsin 0\right)$

$= 4\left(\frac{\pi}{4} - 0\right) = \boxed{\pi}$

ex. $\int_1^{18} \sqrt{\frac{3}{z}}\,dz = \int_1^{18} \frac{\sqrt{3}}{\sqrt{z}}\,dz$

$= \sqrt{3}\int_1^{18} \frac{1}{z^{1/2}}\,dz$

$= \sqrt{3}\int_1^{18} z^{-1/2}\,dz$

$= \underbrace{\sqrt{3}\;\frac{2}{1}}\, z^{1/2}\Big|_1^{18}$

$= 2\sqrt{3}\,\sqrt{z}\Big|_1^{18}$

$= 2\sqrt{3}\left(\sqrt{18} - \sqrt{1}\right)$

$= 2\sqrt{3}\left(3\sqrt{2} - 1\right)$

put in ax^n format

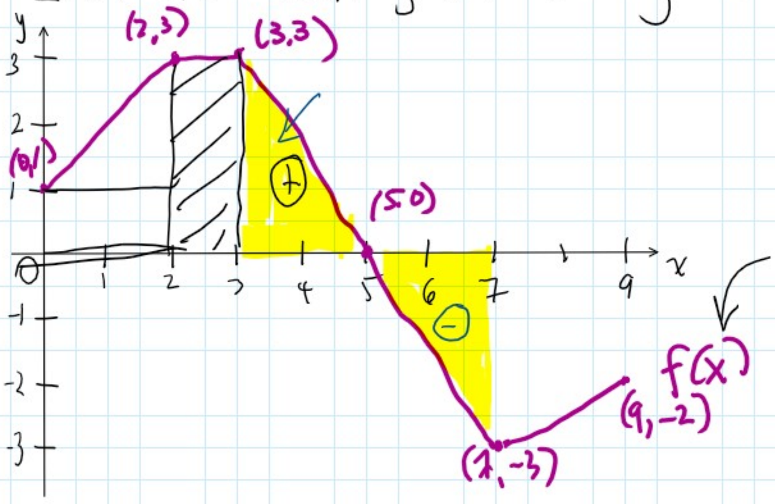
$-\frac{1}{2} + 1\cdot\frac{2}{2} = \frac{1}{2}$

$\int z^{-1/2}\,dz$

$\frac{z^{1/2}}{\textcircled{1/2}} = 2z^{1/2}$

$\sqrt{18} = \sqrt{9}\cdot\sqrt{2}$
$= 3\sqrt{2}$

Evaluate Integrals Using Standard Shapes (revisited)



$\int_0^2 f(x)\,dx = 1(\cancel{2}) + \frac{1}{2}\cancel{(2)}(2)$

$A_{\square}$ $A_{\triangle}$

$= 2 + 2 = 4$

$\int_0^3 f(x)\,dx = \underbrace{\int_0^2 f(x)\,dx}_{4} + \int_2^3 f(x)\,dx$

$+ 1(3)$

$= 7$

$\int_3^7 f(x)\,dx = A_{\triangle} + A_{\triangledown}$

$= \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

$$= \int_3^5 f(x)dx \overset{-}{\cancel{+}} \int_5^7 f(x)\,dx$$

$$\int_3^7 f(x)\,dx = \overset{+}{0}$$

Section 5.4 Fundamental Theorem of Calculus

recall:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)\,dx = 0$$

Fundamental Theorem of Calculus (FTC) establishes a connection between differential calculus and integral calculus

derivatives integrate

These are inverse processes!

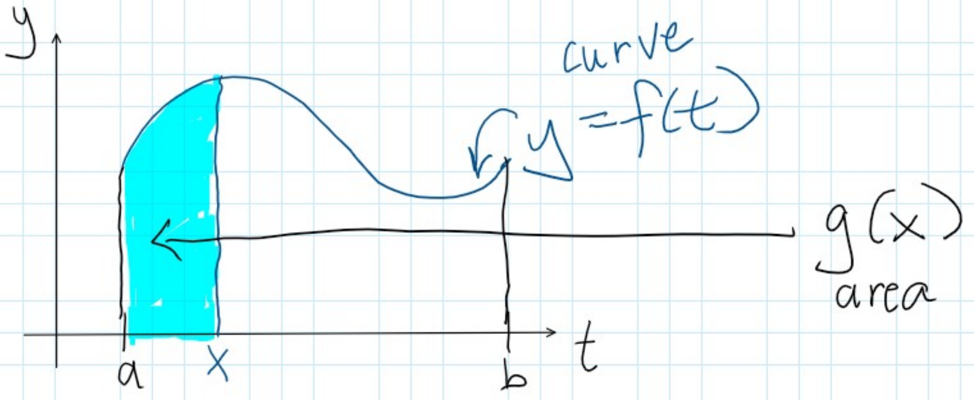
FTC Part 1: $\int_{\underset{\text{constant}}{a}}^{\overset{\text{variable}}{x}} f(t)\,dt = g(x)$ where f is continuous on $[a,b]$

closed interval

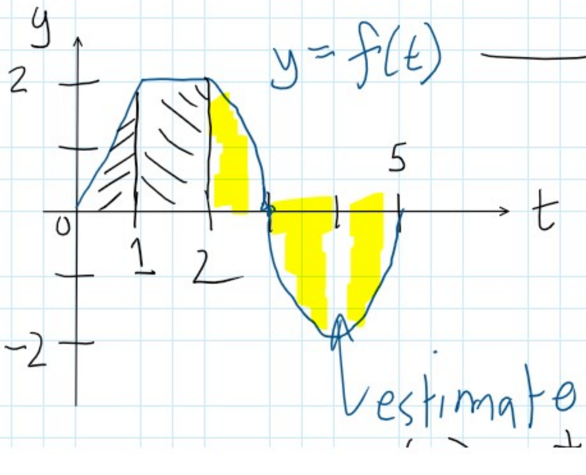
If f is a POSITIVE function then $g(x)$ can be interpreted as the area under the graph of f from a to x where x can vary from a to b.

ie, $g(x)$ is an antiderivative of f

$g'(x) = f$ for $a < x < b$



g(x) area spanned ... so far



$y = f(t) \longrightarrow$ graph represents $g(x) = \int_0^x f(t)\,dt$

evaluate $g(0) = \int_0^0 f(t)\,dt = \boxed{0}$

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}(1)(2) = \boxed{1}$

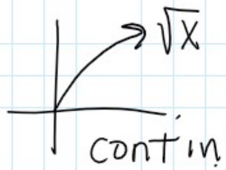
$g(2) = \int_0^2 f(t)\,dt$

$\int^3 f(t)\,dt$

$$\overset{\text{estimate}}{\overbrace{g(2)}} + \int_2^3 f(t)\,dt$$

$$g(3) = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt$$
$$= 3 + \sim 1.3$$
$$= \boxed{4.3}$$

$$g(2) = \int_0 f(t)\,dt$$
$$= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$
$$= 1 + 1(2)$$
$$\rightarrow = \boxed{3}$$



continuous on its domain

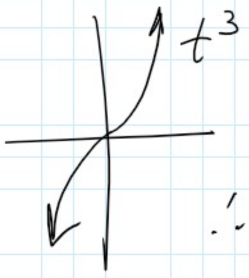
ex. find the <u>derivative</u> of $g(x) = \int_0^x \underbrace{\sqrt{1+t^2}}_{f(t)}\,dt$

FTC Part 1:
$\int_a^x f(t)dt = g(x)$
where f is continuous

$f(t)$ is continuous $t \geq 0$

then by FTC Part 1:

$$g(x) = \sqrt{1+x^2}\;(\underset{1}{x'}) = \boxed{\sqrt{1+x^2}}$$



t^3 is continuous everywhere

$\therefore 7t^3$ is too

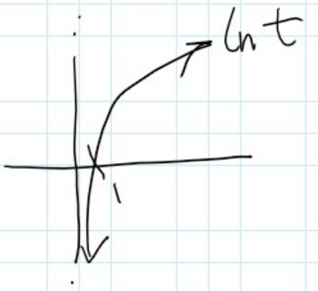
ex. find $S'(x)$ when $S(x) = \int_0^x 7t^3\,dt$

$$= 7x^3\,\boxed{\cancel{(x')}^{1}}$$
$$= \boxed{7x^3}$$

Chain Rule

Applying Chain Rule:

when upper bound is <u>not</u> a simple variable of x, chain rule is necessary



continuous on $t \geq 1$

ex. find $\frac{d}{dx}\int_1^{x^4} \ln t\,dt$

can be <u>any</u> constant - ignore

Let $x^4 = u \rightarrow \frac{du}{dx} = 4x^3$

$$\frac{d}{dx}\int_1^u \ln t\,dt$$

$$\frac{d}{dx} = \frac{d}{du}\cdot\frac{du}{dx}$$

$$\underbrace{\frac{d}{du}\int_1^u \ln t\,dt}_{\ln u}\cdot\frac{du}{dx}$$

$$\ln u \cdot \frac{du}{dx} \equiv \ln u \cdot 4x^3$$
$$\ln x^4 \cdot 4x^3 \quad \text{or} \quad 4x^3 \ln x^4$$

without u-sub:

ex. find $\frac{d}{dx}\int_7^{x^2}(3+t^2)\,dt$

$$= \left(3+(x^2)^2\right)(x^2)'$$
$$= \boxed{(3+x^4)\;2x} \quad \text{or} \quad \boxed{2x\,(3+x^4)}$$

$= \boxed{(3+x^4)\,2x}$ or $\boxed{2x\,(3+x^4)}$

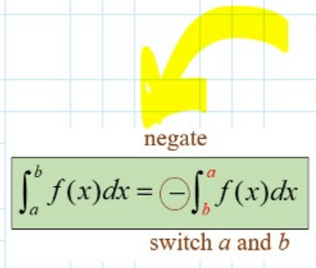
when variable is in lower bound:

ex. find derivative of $\int_{x}^{\pi} \sqrt{1+e^t}\,dt$

$= -\int_{\pi}^{x} \sqrt{1+e^t}\,dt$

↑ ignore once it's lower bound (π is a constant!)

$= \boxed{-\sqrt{1+e^x}}$



negate

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

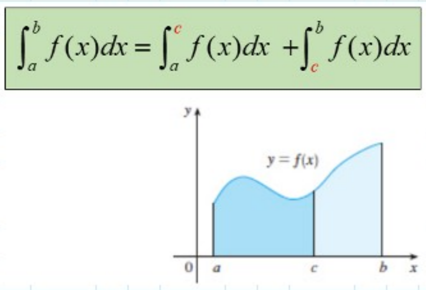
switch a and b

when variable is in both bounds:

both bounds contain variables !!

sol'n: split into 2 integrals

ex. find derivative of $\int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$



$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$= \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du$

$= -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du$ now ready to apply FTC Part 1

$= -\frac{(2x)^2-1}{(2x)^2+1} \cdot 2 + \left[\frac{(3x)^2-1}{(3x)^2+1} \cdot 3\right]$ finally, simplify

$= \boxed{\frac{3(9x^2-1)}{9x^2+1} - \frac{2(4x^2-1)}{4x^2+1}}$

Do: find derivative of each:

$\int_{x^3}^{1} \sqrt{7t^2-3t+6}\,dt$

$\int_{4x}^{9x} \ln t\,dt$

$= \int^{0} \ln t\,dt + \int^{9x} \ln t\,dt$

$$= -\int_1^{x^3} \sqrt{7t^2 - 3t + 6}\ dt$$

$$= -3x^2\sqrt{7(x^3)^2 - 3x^3 + 6}$$

$$= -3x^2\sqrt{7x^6 - 3x^3 + 6}$$

$$= \int_{4x}^{0} \ln t\,dt + \int_0^{9x} \ln t\,dt$$

$$= -\int_0^{4x} \ln t\,dt + 9\ln(9x)$$

$$= -4\ln(4x) + 9\ln(9x)$$

$$= \boxed{9\ln 9x - 4\ln 4x}$$

Complete Fundamental Theorem of Calculus :

Part 1 : if $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$

i.e., $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Part 2 : $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f

that is $F' = f$

i.e. Net Change Theorem :

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$